

On profinite groups in which commutators are covered by finitely many subgroups

Cristina Acciarri and Pavel Shumyatsky

ABSTRACT. For a family of group words w we show that if G is a profinite group in which all w -values are contained in a union of finitely many subgroups with a prescribed property, then $w(G)$ has the same property as well. In particular, we show this in the case where the subgroups are periodic or of finite rank. If G contains finitely many subgroups G_1, G_2, \dots, G_s of finite exponent e whose union contains all γ_k -values in G , it is shown that $\gamma_k(G)$ has finite (e, k, s) -bounded exponent. If G contains finitely many subgroups G_1, G_2, \dots, G_s of finite rank r whose union contains all γ_k -values, it is shown that $\gamma_k(G)$ has finite (k, r, s) -bounded rank.

1. Introduction

A covering of a group G is a family $\{S_i\}_{i \in I}$ of subsets of G such that $G = \bigcup_{i \in I} S_i$. If $\{H_i\}_{i \in I}$ is a covering of G by subgroups, it is natural to ask what information about G can be deduced from properties of the subgroups H_i . In the case where the covering is finite actually quite a lot about the structure of G can be said. The first result in this direction is due to Baer (see [14]), who proved that G admits a finite covering by abelian subgroups if and only if it is central-by-finite. The nontrivial part of this assertion is an immediate consequence of a subsequent result of B.H. Neumann [15]: if $\{S_i\}$ is a finite covering of G by cosets of subgroups, then G is also covered by the cosets S_i corresponding to subgroups of finite index in G . In other words, we can get rid of the cosets of subgroups of infinite index without losing the covering property.

2010 *Mathematics Subject Classification.* 20E18, 20F50.

Key words and phrases. Profinite groups, coverings, verbal subgroups, commutators.

Supported by CAPES and CNPq-Brazil.

Given a group word $w = w(x_1, \dots, x_n)$, we think of it primarily as a function of n variables defined on any given group G . We denote by $w(G)$ the verbal subgroup of G generated by the values of w . If the set of all w -values in a group G can be covered by finitely many subgroups, one could hope to get some information about the structure of the verbal subgroup $w(G)$.

In this direction we mention the following result that was obtained in [17]. Let w be either the lower central word γ_k or the derived word δ_k . Suppose that G is a group in which all w -values are contained in a union of finitely many Chernikov subgroups. Then $w(G)$ is Chernikov.

Another result of this nature was established in [5]: If G is a group in which all commutators are contained in a union of finitely many cyclic subgroups, then G' is either cyclic or finite.

In the present paper we deal with profinite groups in which all w -values are contained in a union of finitely many subgroups with certain prescribed properties. A profinite group is a topological group that is isomorphic to an inverse limit of finite groups. The textbooks [16] and [22] provide a good introduction to the theory of profinite groups. In the context of profinite groups all the usual concepts of group theory are interpreted topologically. In particular, by a subgroup of a profinite group we mean a closed subgroup. A subgroup is said to be generated by a set S if it is topologically generated by S .

The words w considered in this paper are the so-called outer (multilinear) commutator words. Recall that an outer commutator word is a word which is obtained by nesting commutators but using always different variables. A word of this kind has a form of a multilinear Lie monomial. For example the word $[[x_1, x_2], [y_1, y_2, y_5], z]$ is outer commutator while the Engel word $[x_1, x_2, x_2, x_2]$ is not. The word $w(x) = x$ in one variable is an outer commutator word of weight 1; if u and v are defined outer commutator words of weights m and n respectively then, the word $w = [u, v]$ is an outer commutator word of weight $m + n$. An important family of outer commutator words are the lower central words γ_k , given by

$$\gamma_1 = x_1, \quad \gamma_k = [\gamma_{k-1}, x_k] = [x_1, \dots, x_k], \quad \text{for } k \geq 2.$$

The corresponding verbal subgroups $\gamma_k(G)$ are the terms of the lower central series of G . Another distinguished sequence of outer commutator words are the derived words δ_k , on 2^k variables, which are defined recursively by

$$\delta_0 = x_1, \quad \delta_k = [\delta_{k-1}(x_1, \dots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \dots, x_{2^k})], \quad \text{for } k \geq 1.$$

The verbal subgroup that corresponds to the word δ_k is the familiar k th derived subgroup of G usually denoted by $G^{(k)}$.

In the next section we consider profinite groups G having finitely many periodic subgroups G_1, G_2, \dots, G_s whose union contains all w -values in G for some outer commutator word w . Recall that a group is periodic (torsion) if every element of the group has finite order. A group is called locally finite if each of its finitely generated subgroups is finite. It is immediate from the definitions that every locally finite group is periodic. However the converse in general is not true – there exist finitely generated infinite periodic groups (see for example Gupta’s essay [7]). Periodic profinite groups have received a good deal of attention in the past. In particular, using Wilson’s reduction theorem [21], Zelmanov has been able to prove local finiteness of periodic compact groups [23]. Earlier Herfort showed that there exist only finitely many primes dividing the orders of elements of a periodic profinite group [8]. It is a long-standing problem whether any periodic profinite group has finite exponent. Recall that a group is said to be of exponent e if $x^e = 1$ for all $x \in G$ and e is the least positive integer with that property.

The following theorem is the main result of the next section.

THEOREM 1.1. *Let w be an outer commutator word and G a profinite group that has finitely many periodic subgroups G_1, G_2, \dots, G_s whose union contains all w -values in G . Then $w(G)$ is locally finite.*

It follows from the proof that if under the hypothesis of the above theorem the subgroups G_1, G_2, \dots, G_s have finite exponent, then $w(G)$ has finite exponent as well. In Section 3 we address the question whether the exponent of $w(G)$ is bounded in terms of the exponents of G_1, G_2, \dots, G_s and s . Using the Lie-theoretic techniques that Zelmanov created in his solution of the restricted Burnside problem [24], we obtained the following related result.

THEOREM 1.2. *Let e, k, s be positive integers and G a profinite group that has subgroups G_1, G_2, \dots, G_s whose union contains all γ_k -values in G . Suppose that each of the subgroups G_1, G_2, \dots, G_s has finite exponent dividing e . Then $\gamma_k(G)$ has finite (e, k, s) -bounded exponent.*

In Section 4 we study the case where all w -values are contained in a union of finitely many subgroups of finite rank. A group G is said to be of finite rank r if every finitely generated subgroup of G can be generated by r elements. The arguments used in the proofs of Theorems 1.1 and 1.2 turned out to be useful for the case of finite rank. Thus, we obtain the following theorems.

THEOREM 1.3. *Let w be an outer commutator word and G a profinite group that has finitely many subgroups G_1, G_2, \dots, G_s whose union contains all w -values in G . If each of the subgroups G_1, G_2, \dots, G_s is of finite rank, then $w(G)$ has finite rank as well.*

THEOREM 1.4. *Let k, r, s be positive integers and G a profinite group that has subgroups G_1, G_2, \dots, G_s whose union contains all γ_k -values in G . Suppose that each of the subgroups G_1, G_2, \dots, G_s has finite rank at most r . Then $\gamma_k(G)$ has finite (k, r, s) -bounded rank.*

Unlike the situation of Theorem 1.2, the proof of Theorem 1.4 does not use Zelmanov's Lie-theoretic techniques. Instead, an important rôle in the proof is played by the Lubotzky–Mann theory of powerful p -groups [11].

Throughout the paper the expression “ (a, b, \dots) -bounded” stands for “bounded from above by a function depending only on the parameters a, b, \dots ”.

2. Local finiteness of verbal subgroups

Our goal in this section is to prove Theorem 1.1. Zelmanov's theorem that every periodic compact group is locally finite will be used without explicit references. We start with some technical lemmas.

LEMMA 2.1. *Let G be a group, k a positive integer and h, a_1, \dots, a_{2^k} elements of G . Let us denote by x_j and y_j the two elements of G obtained by replacing in $\delta_k(a_1, \dots, a_{2^k})$ the entry a_j with $a_j h$ and h , respectively. Then there exist $h_1, \dots, h_{2^k} \in \langle h^G \rangle$ such that*

$$x_j = \delta_k(a_1^{h_1}, \dots, a_{2^k}^{h_{2^k}}) y_j.$$

PROOF. We argue by induction on k . Assume first that $k = 1$. Denote $a_1^{-1} h a_1$ by t . Using the well-known commutator identities write

$$x_1 = [a_1 h, a_2] = [a_1^h, a_2^h] [h, a_2] = [a_1^h, a_2^h] y_1$$

and

$$x_2 = [a_1, a_2 h] = [a_1, h] [a_1, a_2]^h = [a_1^t, a_2^t] y_2.$$

This shows that if $k = 1$, the lemma is correct. Hence, we assume that $k \geq 2$ and use induction on k . Without loss of generality we also assume that $j \geq 2^{k-1}$. Let us denote by r_j the element of G obtained by replacing in $\delta_{k-1}(a_{2^{k-1}+1}, \dots, a_{2^k})$ the entry a_j with $a_j h$ and by s_j the element of G obtained by replacing in $\delta_{k-1}(a_{2^{k-1}+1}, \dots, a_{2^k})$ the entry a_j with h . By the inductive hypothesis there exist $g_1, \dots, g_{2^{k-1}} \in \langle h^G \rangle$ such that

$$(1) \quad r_j = \delta_{k-1}(a_{2^{k-1}+1}^{g_1}, \dots, a_{2^k}^{g_{2^{k-1}}}) s_j.$$

For the sake of simplicity we denote $\delta_{k-1}(a_1, \dots, a_{2^{k-1}})$ by Δ and observe that $x_j = [\Delta, r_j]$ and $y_j = [\Delta, s_j]$. So by (1) we have

$$\begin{aligned} x_j = [\Delta, r_j] &= [\Delta, \delta_{k-1}(a_{2^{k-1}+1}^{g_1}, \dots, a_{2^k}^{g_{2^{k-1}}})s_j] = \\ &= [\Delta, \delta_{k-1}(a_{2^{k-1}+1}^{g_1}, \dots, a_{2^k}^{g_{2^{k-1}}})]^{s_j y_j^{-1}} [\Delta, s_j]. \end{aligned}$$

Since it is clear that both elements s_j and y_j belong to $\langle h^G \rangle$, the lemma follows. \square

Let G be a group and $w = w(x_1, \dots, x_n)$ a nontrivial group word. Suppose that G has a subgroup H and elements g_1, \dots, g_n such that $w(g_1 h_1, \dots, g_n h_n) = 1$ for all $h_1, h_2, \dots, h_n \in H$. Then we say that the law $w \equiv 1$ is satisfied on the cosets $g_1 H, g_2 H, \dots, g_n H$.

LEMMA 2.2. *Let G be a group and H a normal subgroup of G . Let k be a positive integer and suppose that a_1, \dots, a_{2^k} are elements of G such that the law $\delta_k \equiv 1$ is satisfied on the cosets $a_1 H, \dots, a_{2^k} H$. Then H is soluble with derived length at most k .*

PROOF. By the hypothesis all elements of the form $\delta_k(b_1, \dots, b_{2^k})$, where each entry b_i belongs to $a_i H$, are trivial. Let y be an element that can be obtained from $\delta_k(b_1, \dots, b_{2^k})$ by replacing some of the entries b_1, \dots, b_{2^k} with elements of H . We wish to show that $y = 1$. Suppose that y is obtained from $\delta_k(b_1, \dots, b_{2^k})$ by replacing precisely n of the entries and argue by induction on n . If $n = 0$, then $y = \delta_k(b_1, \dots, b_{2^k}) = 1$. Thus, assume that $n \geq 1$ and use induction on n . Suppose now that $y = \delta_k(c_1, \dots, c_{2^k})$, where each c_i equals either $b_i \in a_i H$ or $h_i \in H$, and choose an index j such that $c_j = h_j$.

Let x be the element obtained from $\delta_k(c_1, \dots, c_{j-1}, a_j, c_{j+1}, \dots, c_{2^k})$ by replacing a_j with $a_j h_j$. Furthermore let us observe that y equals the element obtained from $\delta_k(c_1, \dots, c_{j-1}, a_j, c_{j+1}, \dots, c_{2^k})$ by replacing the entry a_j with h_j . Applying Lemma 2.1 to $\delta_k(c_1, \dots, a_j, \dots, c_{2^k})$ with $h = h_j$ we conclude that there exist $g_1, \dots, g_{2^k} \in H$ such that

$$(2) \quad x = \delta_k(c_1^{g_1}, \dots, a_j^{g_j}, \dots, c_{2^k}^{g_{2^k}})y.$$

Since both x and $\delta_k(c_1^{g_1}, \dots, a_j^{g_j}, \dots, c_{2^k}^{g_{2^k}})$ can be obtained from elements of type $\delta_k(b_1, \dots, b_{2^k})$ by replacing $n-1$ entries with elements of H , by induction we have $x = 1$ and $\delta_k(c_1^{g_1}, \dots, a_j^{g_j}, \dots, c_{2^k}^{g_{2^k}}) = 1$. Thus by (2) also y must be 1.

In the particular case where $n = 2^k$ we have $\delta_k(h_1, \dots, h_{2^k}) = 1$ for all $h_1, \dots, h_{2^k} \in H$. It follows that H is soluble with derived length at most k , as desired. \square

In the next lemma we shall require the concept of the marginal subgroup corresponding to a word. Let G be a group and $w = w(x_1, \dots, x_n)$ any word. The marginal subgroup $w^*(G)$ of G corresponding to the word w is defined as the set of all $a \in G$ such that

$$w(g_1, \dots, ag_i, \dots, g_n) = w(g_1, \dots, g_i, \dots, g_n),$$

for all $g_1, \dots, g_n \in G$ and $1 \leq i \leq n$. It is well known that $w^*(G)$ is a characteristic subgroup of G and that $[w^*(G), w(G)] = 1$. If w is an outer commutator word, then $w^*(G)$ is precisely the set S such that $w(g_1, \dots, g_n) = 1$ whenever at least one of the elements g_1, \dots, g_n belongs to S . A proof of this can be found in [20, Theorem 2.3]. The next lemma was obtained in discussions of the first author with G.A. Fernández-Alcober.

LEMMA 2.3. *Let G be a group and w any outer commutator word. If N is a normal subgroup of G containing no nontrivial w -values, then $[N, w(G)] = 1$.*

PROOF. Let k be the weight of w . Since N is normal and w is an outer commutator word, it follows that whenever $x \in N$ every element of the form

$$w(g_1, \dots, g_{i-1}, x, g_{i+1}, \dots, g_k)$$

belongs to N . Thus, by the hypothesis it must be trivial. Therefore $N \leq w^*(G)$. The result is now clear since $w^*(G)$ always commutes with $w(G)$. \square

A proof of the next lemma can be found in [19, Lemma 4.1].

LEMMA 2.4. *Let G be a group and w an outer commutator word of weight k . Then every δ_k -value in G is a w -value.*

LEMMA 2.5. *Let G be a soluble-by-finite profinite group and suppose that $G/Z(G)$ is periodic. Then G' has finite exponent. In particular G' is locally finite.*

PROOF. It is well-known that a soluble periodic profinite group has finite exponent. For abelian groups this is Exercise 10(c) in [22, page 45] and in the general case the result follows easily by induction on the derived length. Thus, the exponent of $G/Z(G)$ is finite. We denote this by e . A theorem of Mann says that if K is a finite group such that $K/Z(K)$ is of exponent e , then the exponent of K' is e -bounded [13]. Applying now a profinite version of this theorem we conclude that the exponent of G' is finite. \square

PROPOSITION 2.6. *Let G be a profinite group and w any outer commutator word. Suppose that G has finitely many periodic subgroups whose union contains all w -values in G . If G contains an open soluble subgroup, then $w(G)$ is locally finite.*

PROOF. Let d be the minimal among derived lengths of the open normal soluble subgroups of G . If $d = 0$, then G is finite and there is nothing to prove. So we assume that $d \geq 1$ and use induction on d . Thus, G is infinite and let us choose an open normal soluble subgroup K in G such that the derived length of K is precisely d . Let N be the last nontrivial term of the derived series of K . Denote by M the subgroup generated by all w -values that belong to N and let G_1, G_2, \dots, G_s be the finitely many periodic subgroups whose union contains all w -values in G . We have $M = M_1 \cdots M_s$, where $M_i = M \cap G_i$, for $i = 1, \dots, s$. Each subgroup M_i is locally finite, since it is a periodic profinite group, and so it follows that also M is locally finite. We can pass to the quotient G/M and assume that N contains no nontrivial w -values. By Lemma 2.3 we have $w(G) \cap N \leq Z(w(G))$. By induction on d we can assume that the image of $w(G)$ in G/N is locally finite and so $w(G)/Z(w(G))$ is locally finite. Lemma 2.5 now shows that the derived subgroup of $w(G)$ is locally finite. Thus, passing to the quotient $G/w(G)'$ we may assume that $w(G)$ is abelian. Then of course $w(G)$ is the product of subgroups $w(G) \cap G_i$ for $i = 1, \dots, s$. Since $w(G)$ is an abelian group which is a product of finitely many periodic subgroups, it follows that $w(G)$ is locally finite. The proof is complete. \square

We are now in the position to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let k be the weight of w . According to Lemma 2.4 every δ_k -value is also a w -value. For each integer $i = 1, \dots, s$ we set

$$S_i = \left\{ (x_1, \dots, x_{2^k}) \in \underbrace{G \times \cdots \times G}_{2^k} \mid \delta_k(x_1, \dots, x_{2^k}) \in G_i \right\}.$$

Note that the sets S_i are closed in $\underbrace{G \times \cdots \times G}_{2^k}$ and cover the whole of

$\underbrace{G \times \cdots \times G}_{2^k}$. By Baire's Category Theorem [9, p. 200] at least one of

these sets contains a non-empty interior. Hence, there exist an open subgroup H of G , elements a_1, \dots, a_{2^k} in G and an integer j such that

$$(3) \quad \delta_k(a_1 h_1, \dots, a_{2^k} h_{2^k}) \in G_j \text{ for all } h_1, \dots, h_{2^k} \in H.$$

Without loss of generality we can assume that the subgroup H is normal. In this case H normalizes the set of all commutators of the form

$\delta_k(a_1 h_1, \dots, a_{2^k} h_{2^k})$, where $h_1, \dots, h_{2^k} \in H$. Let K be the subgroup of G generated by all commutators of the form $\delta_k(a_1 h_1, \dots, a_{2^k} h_{2^k})$, where $h_1, \dots, h_{2^k} \in H$. Note that $K \leq G_j$. Since the subgroup G_j is locally finite, so is K . Let $D = K \cap H$. Then D is a normal locally finite subgroup of H and the normalizer of D in G has finite index. Therefore there are only finitely many conjugates of D in G . Let $D = D_1, D_2, \dots, D_r$ be all these conjugates. All of them are normal in H and so their product $D_1 D_2 \cdots D_r$ is locally finite. By passing to the quotient $G/D_1 D_2 \cdots D_r$ we may assume that $D = 1$. Since $D = K \cap H$ and H has finite index in G , it follows that K is finite. On the other hand, the normalizer of K has finite index in G and so the normal closure, say L , of K in G is also finite. We can pass to the quotient group G/L and assume that $K = 1$. In that case we have $\delta_k(a_1 h_1, \dots, a_{2^k} h_{2^k}) = 1$ for all $h_1, \dots, h_{2^k} \in H$. Now by Lemma 2.2 the subgroup H is soluble and so the theorem follows immediately from Proposition 2.6. \square

3. Bounding the exponent of $\gamma_k(G)$

The aim of this section is to prove Theorem 1.2. In the next lemma we call a subset X of a group commutator-closed to mean that $[x, y] \in X$ whenever $x, y \in X$. It is clear that the set of all γ_k -values is commutator-closed in any group.

LEMMA 3.1. *Let G be a nilpotent group generated by a commutator-closed subset X which is contained in a union of finitely many subgroups G_1, G_2, \dots, G_s . Then G can be written as the product $G = G_1 G_2 \cdots G_s$.*

PROOF. We use induction on the nilpotency class c of G . If G is abelian, the lemma is clear so we assume that c is at least 2. Let $K = \gamma_c(G)$ and set $K_i = K \cap G_i$ for $i = 1, 2, \dots, s$. Then K is central in G and is generated by commutators in elements of X . It follows that $K = K_1 K_2 \cdots K_s$. By induction we assume that

$$G = G_1 G_2 \cdots G_s K = G_1 G_2 \cdots G_s K_1 K_2 \cdots K_s.$$

Since the subgroups K_1, K_2, \dots, K_s are central and $K_i \leq G_i$, we deduce that $G = G_1 G_2 \cdots G_s$, as required. \square

Let P be a Sylow p -subgroup of a finite group G . An immediate corollary of the Focal Subgroup Theorem [3] is that $P \cap G'$ is generated by commutators. We do not know if $P \cap w(G)$ is generated by w -values for every outer commutator word w . A proof of the next related result can be found in [1, Theorem A].

THEOREM 3.2. *Let G be a finite group, p a prime, and P a Sylow p -subgroup of G . If w is an outer commutator word, then $P \cap w(G)$ is generated by powers of w -values.*

In this section the above theorem is applied in the case where w is a lower central word.

The routine inverse limit argument shows that Theorem 1.2 can be easily deduced from the corresponding result for finite groups. Thus, we will deal with the case where G is finite. In the proof of the next theorem we use a result from [18] whose proof is based on Zelmanov's Lie-theoretic techniques.

THEOREM 3.3. *Let e, k, s be positive integers. Let G be a finite group that has subgroups G_1, G_2, \dots, G_s whose union contains all γ_k -values of G . Suppose that each subgroup G_i has finite exponent dividing e . Then $\gamma_k(G)$ has finite exponent bounded by a function depending on e, k and s only.*

PROOF. Let P be a Sylow p -subgroup of $\gamma_k(G)$. It is sufficient to show that the exponent of P is (e, k, s) -bounded. Theorem 3.2 shows that P is generated by elements of order dividing e . Therefore it is sufficient to show that the exponent of $\gamma_k(P)$ is (e, k, s) -bounded. Thus, we may assume from the beginning that G is a p -group. Further, without loss of generality we may assume that each subgroup G_i is generated by γ_k -values and so $\gamma_k(G) = \langle G_1, G_2, \dots, G_s \rangle$. It follows from Lemma 3.1 that $\gamma_k(G) = G_1 G_2 \cdots G_s$. We wish to prove that an arbitrary element of $\gamma_k(G)$ has bounded order. Thus, choose $g \in \gamma_k(G)$. We know that there exist elements $g_i \in G_i$ for $i = 1, 2, \dots, s$ such that $g = g_1 g_2 \cdots g_s$. Set $R = \langle g_1, g_2, \dots, g_s \rangle$. The subgroup R is generated by s elements, each of order dividing e and every γ_k -value in R is of order dividing e as well. By [18, Theorem 7] the order of R is (e, k, s) -bounded. In particular, the order of g is (e, k, s) -bounded, as required. \square

The proof of Theorem 1.2 is now complete.

4. Bounding the rank of a verbal subgroup

In this section we will prove Theorems 1.3 and 1.4. First we will show that Theorem 1.3 holds under the additional assumption that G contains an open soluble subgroup.

PROPOSITION 4.1. *Let G be a profinite group and w any outer commutator word. Suppose that G has finitely many subgroups of finite rank whose union contains all w -values in G . If G contains an open soluble subgroup, then the rank of $w(G)$ is finite.*

PROOF. Arguing as in the proof of Proposition 2.6 let us define d to be the minimum of the derived lengths of the open normal soluble subgroups of G . If $d = 0$, then G is finite and there is nothing to prove. So we use induction on d and assume that $d \geq 1$. In particular G is infinite and choose an open normal soluble subgroup H of G such that the derived length of H is precisely d . Let N be the last nontrivial term of the derived series of H and let M be the subgroup generated by all w -values that belong to N . Let G_1, G_2, \dots, G_s be the finitely many subgroups of finite rank whose union contains all w -values in G . We have $M = M_1 \cdots M_s$, where $M_i = M \cap G_i$ for $i = 1, \dots, s$. Note that each subgroup M_i is of finite rank, since G_i has finite rank by the hypothesis. It follows that M has finite rank as well. We pass to the quotient G/M and assume that N contains no nontrivial w -values. By Lemma 2.3 we have $w(G) \cap N \leq Z(w(G))$. By induction on d we assume that the image of $w(G)$ in G/N is of finite rank and so $w(G)/Z(w(G))$ is of finite rank. Theorem 2.5 in [6] tells us that if K is a soluble-by-finite group such that $K/Z(K)$ has finite rank, then K' has finite rank as well. Moreover the rank of K' is bounded in terms of the derived length of the soluble radical of K , its index in K and the rank of $K/Z(K)$. The profinite version of this result is straightforward and so we are in a position to apply it with $K = w(G)$. It follows that the rank of the derived group of $w(G)$ is finite. Thus, passing to the quotient $G/w(G)'$ we may assume that $w(G)$ is abelian. Then $w(G)$ is a product of finitely many subgroups of finite rank and we conclude that the rank of $w(G)$ is finite, as required. \square

We can now deal with the general case.

PROOF OF THEOREM 1.3. Let k be the weight of w . According to Lemma 2.4 every δ_k -value is also a w -value. For each integer $i = 1, \dots, s$ we set

$$S_i = \left\{ (x_1, \dots, x_{2^k}) \in \underbrace{G \times \cdots \times G}_{2^k} \mid \delta_k(x_1, \dots, x_{2^k}) \in G_i \right\}.$$

The sets S_i are closed in $\underbrace{G \times \cdots \times G}_{2^k}$ and cover the group $\underbrace{G \times \cdots \times G}_{2^k}$.

By Baire's Category Theorem [9, p. 200] at least one of these sets contains a non-empty interior. Hence, there exist an open subgroup H of G , elements a_1, \dots, a_{2^k} in G and an integer j such that

$$\delta_k(a_1 h_1, \dots, a_{2^k} h_{2^k}) \in G_j \text{ for all } h_1, \dots, h_{2^k} \in H.$$

Without loss of generality we can assume that the subgroup H is normal. Then it is clear that H normalizes the set of all commutators of the form $\delta_k(a_1 h_1, \dots, a_{2^k} h_{2^k})$, where $h_1, \dots, h_{2^k} \in H$. Let K be the subgroup of G generated by all commutators of the form $\delta_k(a_1 h_1, \dots, a_{2^k} h_{2^k})$, where $h_1, \dots, h_{2^k} \in H$. Note that $K \leq G_j$. Since the rank of G_j is finite by the hypothesis, it follows that also K has finite rank. Let $D = K \cap H$. Then D is a normal subgroup of finite rank in H and the normalizer of D in G has finite index. Therefore there are only finitely many conjugates of D in G . Let $D = D_1, D_2, \dots, D_r$ be all these conjugates. All of them are normal in H and so their product $D_1 D_2 \cdots D_r$ has finite rank. By passing to the quotient $G/D_1 D_2 \cdots D_r$ we may assume that $D = 1$. Since $D = K \cap H$ and H has finite index in G , it follows that K is finite. On the other hand, the normalizer of K has finite index in G and so the normal closure, say L , of K in G is also finite. Thus we can pass to the quotient G/L and assume that $K = 1$. In this case we have $\delta_k(a_1 h_1, \dots, a_{2^k} h_{2^k}) = 1$ for all $h_1, \dots, h_{2^k} \in H$. By Lemma 2.2 the subgroup H is soluble and so the theorem follows immediately from Proposition 4.1. \square

We will now proceed to the proof of Theorem 1.4. The result can be easily deduced from the corresponding result on finite groups. Thus we will deal with the case where G is a finite group. We will mention now some results about rank of a finite group. The following theorem is an immediate consequence of a result obtained independently by Guralnick [4] and Lucchini [12]. It depends on the classification of finite simple groups. The result for soluble groups was obtained by Kovács [10].

THEOREM 4.2. *Let G be a finite group in which the rank of every Sylow subgroup is at most r . Then the rank of G is at most $r + 1$.*

The next lemma essentially is due to Lubotzky and Mann [11]. For the reader's convenience we give a proof here.

LEMMA 4.3. *Let P be a finite p -group and suppose that every term of the lower central series of P can be generated by d elements. Then the rank of P is d -bounded.*

PROOF. Let V be the intersection of kernels of all homomorphisms of P into $GL_d(\mathbb{F})$, where \mathbb{F} is the field with p elements. Set $W = V$ if p is odd and $W = V^2$ if $p = 2$. Then any characteristic d -generated subgroup of P contained in W is powerful by Proposition 2.12 in [2]. Since the Sylow p -subgroups of $GL_d(\mathbb{F})$ are nilpotent of class $d - 1$, it follows that $\gamma_d(P) \leq V$. We know that $\gamma_d(P)$ is d -generated so

the image of $\gamma_d(G)$ in P/W has order at most 2^d . Therefore P/W is nilpotent of class at most $2d-1$ whence $\gamma_{2d}(P) \leq W$. Since $\gamma_{2d}(P)$ has at most d generators, it becomes clear that $\gamma_{2d}(P)$ is powerful. Thus we conclude that $\gamma_{2d}(P)$ has rank at most d [2, Theorem 2.9]. Since P has at most d generators, it is easy to see that the rank of $P/\gamma_{2d}(P)$ is d -bounded. The lemma follows. \square

THEOREM 4.4. *Let k, r and s be positive integers. Let G be a finite group that has subgroups G_1, G_2, \dots, G_s whose union contains all γ_k -values of G . Suppose that each subgroup G_i is of rank at most r . Then the rank of $\gamma_k(G)$ is bounded by a function depending on k, r and s only.*

PROOF. We remark that our hypothesis implies that if H is a subgroup of G generated by the intersections $H \cap G_i$, then H can be generated by at most rs elements. Hence, for every $j \geq k$ and every subgroup U of G the corresponding term of the lower central series $\gamma_j(U)$ can be generated by at most rs elements. It follows from Lemma 4.3 that every p -subgroup with rs generators has $\{k, r, s\}$ -bounded rank. Now let P be a Sylow p -subgroup of $\gamma_k(G)$. Theorem 3.2 tells us that P is generated by powers of γ_k -values. Obviously the powers are contained in the union of the subgroups G_i . Hence P can be generated by at most rs elements and so the rank of P is $\{k, r, s\}$ -bounded. Now the theorem is immediate from Theorem 4.2. \square

The proof of Theorem 1.4 is now complete.

References

- [1] C. Acciarri, G.A. Fernández-Alcober, P. Shumyatsky, A focal subgroup theorem for outer commutator words. arXiv:1108.2284.
- [2] J.D. Dixon, M.P.F. du Sautoy, A. Mann, D. Segal, *Analytic Pro- p Groups*, 2nd Edition, Cambridge University Press, Cambridge, 1991.
- [3] D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.
- [4] R. Guralnick, On the number of generators of a finite group, *Arch. Math.*, **53** (1989), 521–523.
- [5] G.A. Fernández-Alcober, P. Shumyatsky, On groups in which commutators are covered by finitely many cyclic subgroups, *J. Algebra* **319** (2008), 4844–4851.
- [6] S. Franciosi, F. de Giovanni, L.A. Kurdachenko, The Schur property and groups with uniform conjugacy classes, *J. Algebra* **174** (1995), 823–847.
- [7] N. Gupta, On groups in which every element has finite order, *The American Mathematical Monthly* **96** (1989), 297–308.
- [8] W. Herfort, Compact torsion groups and finite exponent, *Arch. Math.* **33** (1979), 404–410.
- [9] J.L. Kelley, *General Topology*, van Nostrand, Toronto - New York - London, 1955.

- [10] L. Kovács, On finite soluble groups, *Mathematische Zeitschrift* **103** (1968), 37–39.
- [11] A. Lubotzky, A. Mann, Powerful p -groups. I: finite groups, *J. Algebra* **105** (1987), 484–505.
- [12] A. Lucchini, A bound on the number of generators of a finite group, *Arch. Math.* **53** (1989), 313–317.
- [13] A. Mann, The exponents of central factor and commutator groups, *J. Group Theory* **10** (2007), 435–436.
- [14] B.H. Neumann, Groups covered by finitely many cosets, *Publ. Math. Debrecen* **3** (1954), 227–242.
- [15] B.H. Neumann, Groups covered by permutable subsets, *J. London Math. Soc.* **29** (1954), 236–248.
- [16] L. Ribes, P. Zalesskii, *Profinite Groups*, 2nd Edition, Springer Verlag, Berlin – New York, 2010.
- [17] J.R. Rogério, P. Shumyatsky, A finiteness condition for verbal subgroups, *J. Group Theory* **10** (2007), 811–815.
- [18] P. Shumyatsky, A (locally nilpotent)-by-nilpotent variety of groups, *Math. Proc. Cambridge Philos. Soc.* **132** (2002), 193–196.
- [19] P. Shumyatsky, Verbal subgroups in residually finite groups, *Quart. J. Math.* **51** (2000), 523–528.
- [20] R.F. Turner-Smith, Marginal subgroup properties for outer commutator words, *Proc. London Math. Soc.* **14** (1964), 321–341.
- [21] J.S. Wilson, On the structure of compact torsion groups, *Monatsh. Math.* **96** (1983), 57–66.
- [22] J.S. Wilson, *Profinite Groups*, Clarendon Press, Oxford, 1998.
- [23] E. Zelmanov, On periodic compact groups, *Israel J. Math.* **77** (1992), 83–95.
- [24] E. Zelmanov, Lie methods in the theory of nilpotent groups, in *Groups '93 Galway/ St Andrews*, Cambridge University Press, Cambridge, (1995), 567–585.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRASILIA, BRASILIA - DF,
BRAZIL

E-mail address: acciarriacristina@yahoo.it

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRASILIA, BRASILIA - DF,
BRAZIL

E-mail address: pavel@mat.unb.br